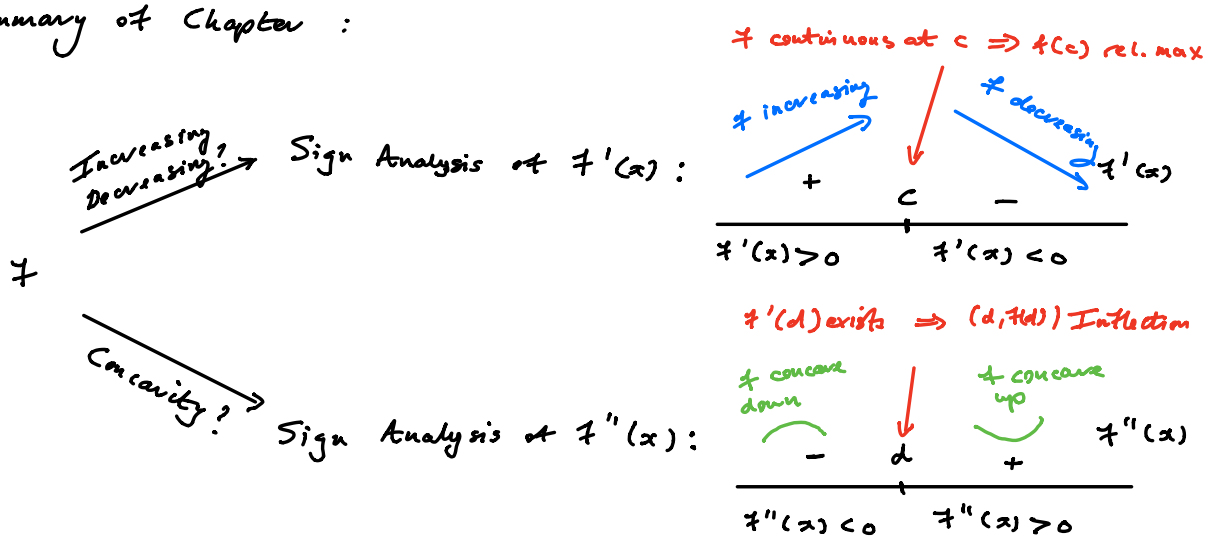
Curve Sketching

Summary of Chapter :



Aim : Put everything together to sketch graph $y = f(x)$.

Strategy :

1/ Determine domain (look for square roots and denominators being 0)

2/ If possible find x and y intercepts.

3/ a) Look for horizontal asymptotes, ie. $\lim_{x \to \pm\infty} f(x) = L \Rightarrow y = L$ a horizontal asymptote

Important examples : Rational and exponential functions

b) Look for vertical asymptotes, ie $\lim_{x \to a^{+/-}} f(x) = \infty \Rightarrow x = a$ vertical asymptotes

Important examples : Rational and logarithmic functions.

$x = 0$ for both $\frac{1}{x}$ and $\ln(x)$

4/ Is function odd / even ?

5/ Determine increasing / decreasing / relative extrema using $f'(x)$.

6/ Determine concavity / inflection points using $f''(x)$

7/ Plot intercepts, critical points, inflection points and asymptotes.

Connect together following sign analysis done in 5/ and 6/

Example different independent variable than usual.

$R(x)$ = revenue generated after spending x dollars on advertising.

Assume $R(x) = \frac{1}{15,000}(600x^2 - x^3)$ $\quad 0 \le x \le 600$

Sketch $y = R(x)$.

1/ Domain = $[0, 600]$

2/ $R(0) = 0$ = y-intercept

$R(x) = 0 \Rightarrow \frac{1}{15000}(600x^2 - x^3) = 0$

$\Rightarrow \frac{1}{15000} x^2 (600 - x) = 0 \Rightarrow x = 0$ or 600

3/ $R(x)$ continuous everywhere on $[0, 600]$ $\Rightarrow$ No asymptotes

4/ Neither odd nor even

5/ $R'(x) = \frac{1}{15000}(1200x - 3x^2)$

A/ $R'(x) = 0 \Rightarrow \frac{1}{15000}(1200x - 3x^2) = 0$

$\Rightarrow \frac{1}{15000} x (1200 - 3x) = 0 \Rightarrow x = 0$ or 400

B/ R' continuous everywhere

$R(400)$ defined $\Rightarrow$ $R(400)$ relative max

$R'(100) = \frac{1}{15000} 100 (1200 - 3 \cdot 100) > 0$ $\quad R'(500) = \frac{1}{15000} \cdot 500 (1200 - 3 \cdot 500) < 0$

6/ $R''(x) = \frac{1}{15000}(1200 - 6x)$

A/ $R''(x) = 0 \Rightarrow \frac{1}{15000}(1200 - 6x) = 0 \Rightarrow 1200 - 6x = 0 \Rightarrow x = 200$

B/ R'' continuous everywhere.

$R'(200)$ defined $\Rightarrow$ $(200, R(200))$ inflection

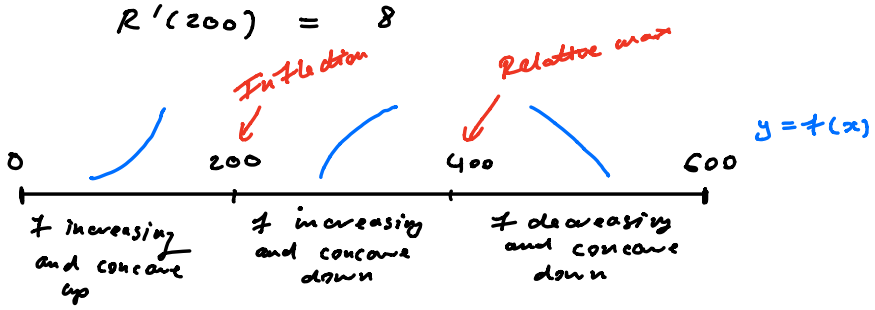
$y''(100) = \frac{1}{15000}(1200 - 600) > 0$ $\quad y''(400) = \frac{1}{15000}(1200 - 6 \cdot 400) < 0$

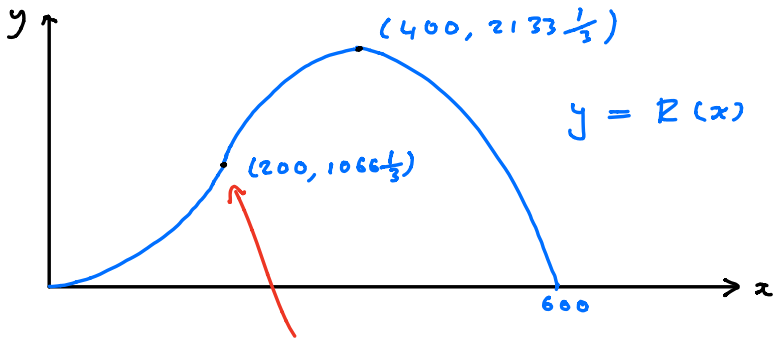
7/ $R(0) = R(600) = 0$

$R(400) = 2133\frac{1}{3}$

$R(200) = 1066\frac{2}{3}$

$R'(200) = 8$





Called point of diminishing returns by economists.

Example Sketch $y = \frac{2x+2}{x-1}$ $(= f(x))$

1/ Domain = All $x \neq 1$

2/ $f(0) = \frac{2}{-1} = -2 =$ y-intercept

$f(x) = 0 \Rightarrow 2x+2 = 0 \Rightarrow x = -1 =$ x-intercept

3/ A/ $\lim_{x \to \pm\infty} \frac{2x+2}{x-1} = 2 \Rightarrow y = 2$ horizontal asymptote in both directions.

B/ denominator $(x-1) = 0 \Rightarrow x = 1$, numerator $2 \cdot 1 + 2 = 4 \neq 0$

$\Rightarrow x = 1$ vertical asymptote

4/ Neither odd or even

5/ $f(x) = \frac{u}{v}$, $u = 2x+2$, $v = x-1$

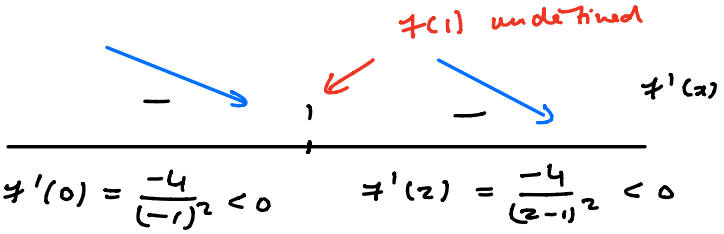
$\Rightarrow u'(x) = 2$, $v'(x) = 1$

$\Rightarrow f'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{(v(x))^2} = \frac{2(x-1) - (2x+2)\cdot 1}{(x-1)^2}$

$= \frac{-4}{(x-1)^2}$

A/ $f'(x) = 0 \Rightarrow \frac{-4}{(x-1)^2} = 0$ (No solutions)

B/ f' undefined at $x = 1$



$f'(0) = \frac{-4}{(-1)^2} < 0$ $\quad f'(2) = \frac{-4}{(2-1)^2} < 0$

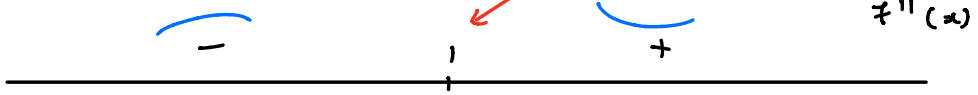
6/ $f''(x) = \frac{d}{dx}\left(\frac{-4}{(x-1)^2}\right) = -4\frac{d}{dx}(x-1)^{-2}$ ← Chain Rule

$= (-4)\cdot(-2)(x-1)^{-3}$

$= \frac{8}{(x-1)^3}$

A/ $f''(x) = 0 \Rightarrow \frac{8}{(x-1)^3} = 0$ (No solutions)

B/ f'' undefined at $x = 1$ $\quad$ $f'(1)$ undefined $\Rightarrow$ Not inflection

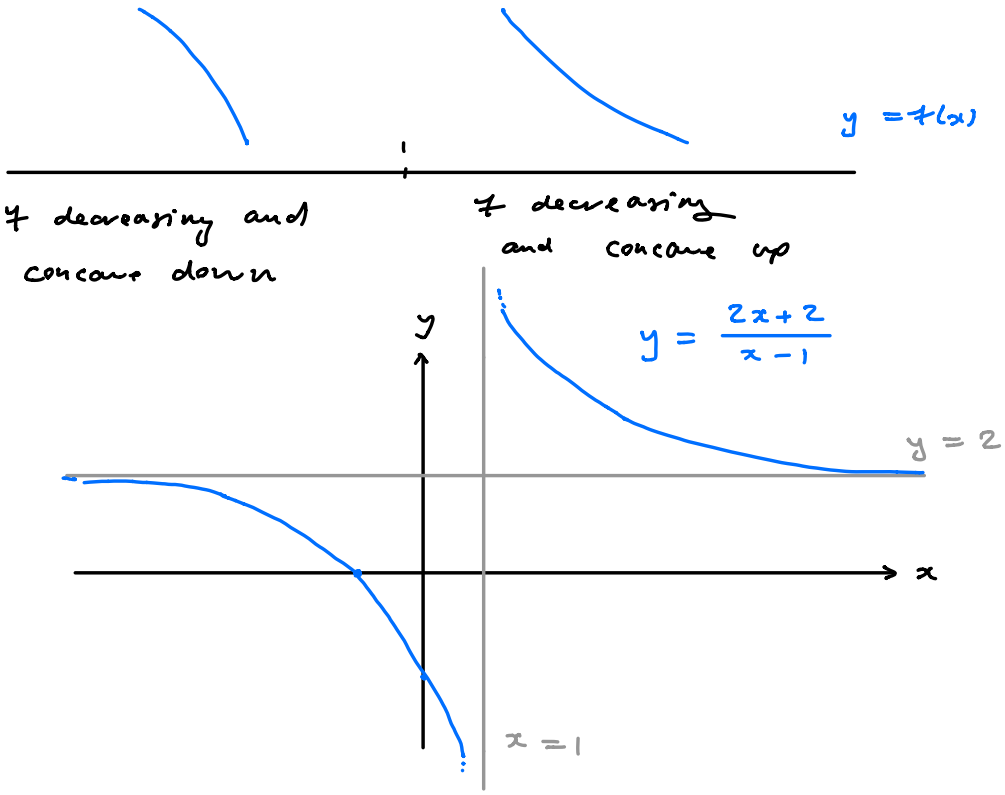


$f''(0) = -8 < 0$ $\quad f''(2) = 8 > 0$

7/ No critical points or inflections.

x-intercept : $(-1, 0)$

y-intercept : $(0, -2)$


y = f(x)
1
f decreasing and concave down
f decreasing and concave up
y
x
y = (2x + 2)/(x − 1)
y = 2
x = 1